

Weighted representation functions on \mathbb{Z}_m^*

Quan-Hui Yang and Yong-Gao Chen[†]

School of Mathematical Sciences and Institute of Mathematics,

Nanjing Normal University, Nanjing 210046, P. R. CHINA

Abstract

Let m , k_1 , and k_2 be three integers with $m \geq 2$. For any set $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$, let $\hat{r}_{k_1, k_2}(A, n)$ denote the number of solutions of the equation $n = k_1 a_1 + k_2 a_2$ with $a_1, a_2 \in A$. In this paper, using exponential sums, we characterize all m , k_1 , k_2 , and A for which $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. We also pose several problems for further research.

2010 Mathematics Subject Classifications: 11B34

Key words and phrases: exponential sums, representation functions.

1 Introduction

Let \mathbb{N} be the set of nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A, n)$, $R_2(A, n)$, $R_3(A, n)$ denote the number of solutions of $a + a' = n, a, a' \in A$; $a + a' = n, a, a' \in A, a < a'$ and $a + a' = n, a, a' \in A, a \leq a'$ respectively. For

^{*}This work was supported by the National Natural Science Foundation of China, Grant No. 11071121 and the Project of Graduate Education Innovation of Jiangsu Province (CXZZ12-0381).

[†]Corresponding author. Email: yangquanhui01@163.com, ygchen@njnu.edu.cn

$i \in \{1, 2, 3\}$, Sárközy asked ever whether there are sets A and B with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n . It is known that the answer is negative for $i = 1$ (see Dombi [4]) and the answer is positive for $i = 2, 3$ (see Dombi [4], Chen and Wang [3]). In fact, Dombi [4] for $i = 2$ and Chen and Wang [3] for $i = 3$ proved that there exists a set $A \subseteq \mathbb{N}$ such that $R_i(A, n) = R_i(\mathbb{N} \setminus A, n)$ for all $n \geq n_0$. Lev (see [5]) gave a simple common proof to the results by Dombi [4] and Chen and Wang [3]. Finally, using generating functions, Sándor [6] gave a complete answer by using generating functions, and later Tang [7] gave an elementary proof. For related research, one may refer to [1] and [2]. For a positive integer m , let \mathbb{Z}_m be the set of residue classes modulo m . For the modular version, the first author and Chen [8] proved that if and only if m is even, there exists $A \in \mathbb{Z}_m$ such that $R_1(A, n) = R_1(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$.

For any given two positive integers k_1, k_2 and any set A of nonnegative integers, let $r_{k_1, k_2}(A, n)$ denote the number of solutions of the equation $n = k_1 a_1 + k_2 a_2$ with $a_1, a_2 \in A$. Recently, the authors [9] proved that there exists a set $A \subseteq \mathbb{N}$ such that $r_{k_1, k_2}(A, n) = r_{k_1, k_2}(\mathbb{N} \setminus A, n)$ for all sufficiently large integers n if and only if $k_1 \mid k_2$ and $k_2 > k_1$.

For any given t integers k_1, \dots, k_t , and any set $A \subseteq \mathbb{Z}_m$ and $n \in \mathbb{Z}_m$, let $\hat{r}_{k_1, \dots, k_t}(A, n)$ denote the number of solutions of the equation $n = k_1 a_1 + \dots + k_t a_t$ with $a_1, \dots, a_t \in A$. In this paper, we prove the following theorem.

Theorem 1. *Let m, k_1 , and k_2 be three integers with $m \geq 2$, and let $A \subseteq \mathbb{Z}_m$. Then $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if $|A| = m/2$ and A is uniformly distributed modulo $d_1 d_2 / d_3^2$, where $(k_1, m) = d_1$, $(k_2, m) = d_2$, and $(d_1, d_2) = d_3$.*

Corollary 1. *Let m, k_1 , and k_2 be three integers with $m \geq 2$. Then there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$ if and only if m is even and one of the following statements is true:*

- (i) k_1 and k_2 have the same parity;

(ii) k_1 and k_2 have the different parities with $v_2(k_i) < v_2(m)$ ($i = 1, 2$), where $v_2(k) = t$ if $2^t \mid k$ and $2^{t+1} \nmid k$.

Motivated by Lev [5] and the authors [9], we now pose the following problems for further research.

Problem 1. For any given two integers k_1 and k_2 , determine all pairs of subsets $A, B \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(B, n)$ for all $n \in \mathbb{Z}_m$.

Problem 2. For $t \geq 3$, find all $t + 1$ -tuples (m, k_1, \dots, k_t) of integers for which there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1, \dots, k_t}(A, n) = \hat{r}_{k_1, \dots, k_t}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$.

2 Proofs

For $T \subseteq \mathbb{Z}_m$ and $x \in \mathbb{Z}_m$, let

$$S_T(x) = \sum_{t \in T} e^{2\pi i t x / m}.$$

Let $A \subseteq \mathbb{Z}_m$ and $B = \mathbb{Z}_m \setminus A$. Then

$$\hat{r}_{k_1, k_2}(A, n) = \sum_{x=0}^{m-1} S_A(k_1 x) S_A(k_2 x) e^{-2\pi i n x / m}$$

for all $n \in \mathbb{Z}_m$. Let $g_A(x) = S_A(k_1 x) S_A(k_2 x) - S_B(k_1 x) S_B(k_2 x)$. Thus

$$\hat{r}_{k_1, k_2}(A, n) - \hat{r}_{k_1, k_2}(B, n) = \sum_{x=0}^{m-1} g_A(x) e^{-2\pi i n x / m} \quad (1)$$

for all $n \in \mathbb{Z}_m$.

In order to prove Theorem 1, we need the following Lemmas.

Lemma 1. Let m, k_1, k_2 be three integers with $m \geq 2$. If $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(B, n)$ for all $n \in \mathbb{Z}_m$, then m is even and $|A| = m/2$.

Proof. If $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(B, n)$ holds for all $n \in \mathbb{Z}_m$, then we have

$$|A|^2 = \sum_{n \in \mathbb{Z}_m} \hat{r}_{k_1, k_2}(A, n) = \sum_{n \in \mathbb{Z}_m} \hat{r}_{k_1, k_2}(B, n) = |B|^2.$$

Hence we get $|A| = |B|$, that is, m is even and $|A| = m/2$. \square

Lemma 2. *If $m \nmid k_i x$ ($i = 1, 2$), then $g_A(x) = 0$.*

Proof. Since $m \nmid k_i x$ ($i = 1, 2$), it follows that

$$S_A(k_1 x) + S_B(k_1 x) = \sum_{j=0}^{m-1} e^{2\pi i k_1 x j / m} = 0$$

and

$$S_A(k_2 x) + S_B(k_2 x) = \sum_{j=0}^{m-1} e^{2\pi i k_2 x j / m} = 0.$$

Hence $g_A(x) = S_A(k_1 x)S_A(k_2 x) - S_B(k_1 x)S_B(k_2 x) = 0$. □

Lemma 3. *If $|A| = m/2$ and $m \mid k_i x$ ($i = 1, 2$), then $g_A(x) = 0$.*

Proof. Since $m \mid k_i x$ ($i = 1, 2$), it follows that

$$S_A(k_1 x) = |A| = S_A(k_2 x) \quad \text{and} \quad S_B(k_1 x) = |B| = S_B(k_2 x).$$

Thus $g_A(x) = |A|^2 - |B|^2$. By $|A| = m/2$ we have $|B| = m/2$. Therefore, $g_A(x) = 0$. □

Lemma 4. *If k and ℓ are two integers, then*

$$\sum_{\substack{x=0 \\ m \mid kx}}^{m-1} S_T(\ell x) e^{-2\pi i n x / m} = (k, m) \sum_{\substack{t \in T \\ (k, m) \mid \ell t - n}} 1.$$

Proof. Let $d = (k, m)$. Then

$$\begin{aligned} \sum_{\substack{x=0 \\ m \mid kx}}^{m-1} S_T(\ell x) e^{-2\pi i n x / m} &= \sum_{\substack{x=0 \\ m \mid kx}}^{m-1} \sum_{t \in T} e^{2\pi i (\ell t - n) x / m} \\ &= \sum_{s=0}^{d-1} \sum_{t \in T} e^{2\pi i (\ell t - n) s / d} = d \sum_{\substack{t \in T \\ d \mid \ell t - n}} 1. \end{aligned}$$

□

Proof of Theorem 1. By Lemma 1 we may assume that m is even and $|A| = |B| = m/2$. From (1), by Lemmas 2-4, we have

$$\begin{aligned}
& \hat{r}_{k_1, k_2}(A, n) - \hat{r}_{k_1, k_2}(B, n) \\
&= \sum_{\substack{x=0 \\ m \nmid k_1 x, m \nmid k_2 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} + \sum_{\substack{x=0 \\ m \mid k_1 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} \\
&\quad + \sum_{\substack{x=0 \\ m \mid k_2 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} - \sum_{\substack{x=0 \\ m \mid k_1 x, m \mid k_2 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} \\
&= \sum_{\substack{x=0 \\ m \mid k_1 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} + \sum_{\substack{x=0 \\ m \mid k_2 x}}^{m-1} g_A(x) e^{-2\pi i n x / m} \\
&= \frac{m}{2} \sum_{\substack{x=0 \\ m \mid k_1 x}}^{m-1} (S_A(k_2 x) - S_B(k_2 x)) e^{-2\pi i n x / m} \\
&\quad + \frac{m}{2} \sum_{\substack{x=0 \\ m \mid k_2 x}}^{m-1} (S_A(k_1 x) - S_B(k_1 x)) e^{-2\pi i n x / m} \\
&= \frac{1}{2} m d_1 \left(\sum_{\substack{a \in A \\ d_1 \mid k_2 a - n}} 1 - \sum_{\substack{b \in B \\ d_1 \mid k_2 b - n}} 1 \right) + \frac{1}{2} m d_2 \left(\sum_{\substack{a \in A \\ d_2 \mid k_1 a - n}} 1 - \sum_{\substack{b \in B \\ d_2 \mid k_1 b - n}} 1 \right).
\end{aligned}$$

It follows that

$$\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(B, n) \quad (2)$$

is equivalent to

$$d_1 \sum_{\substack{a \in A \\ d_1 \mid k_2 a - n}} 1 + d_2 \sum_{\substack{a \in A \\ d_2 \mid k_1 a - n}} 1 = d_1 \sum_{\substack{b \in B \\ d_1 \mid k_2 b - n}} 1 + d_2 \sum_{\substack{b \in B \\ d_2 \mid k_1 b - n}} 1. \quad (3)$$

Suppose that (2) holds for all $n \in \mathbb{Z}_m$. Then (3) holds for all $n \in \mathbb{Z}_m$.

Thus

$$d_1 \sum_{\substack{a \in A \\ d_1 \mid k_2 a - d_3 n}} 1 + d_2 \sum_{\substack{a \in A \\ d_2 \mid k_1 a - d_3 n}} 1 = d_1 \sum_{\substack{b \in B \\ d_1 \mid k_2 b - d_3 n}} 1 + d_2 \sum_{\substack{b \in B \\ d_2 \mid k_1 b - d_3 n}} 1 \quad (4)$$

for all $n \in \mathbb{Z}_m$. Let

$$d_i = d_3 d'_i, \quad k_i = d_3 k'_i, \quad i = 1, 2.$$

From (4), we have

$$d_1 \sum_{\substack{a \in A \\ d'_1 | k'_2 a - n}} 1 + d_2 \sum_{\substack{a \in A \\ d'_2 | k'_1 a - n}} 1 = d_1 \sum_{\substack{b \in B \\ d'_1 | k'_2 b - n}} 1 + d_2 \sum_{\substack{b \in B \\ d'_2 | k'_1 b - n}} 1. \quad (5)$$

Since $(d_1, k_2) = (k_1, m, k_2) = d_3$, it follows that $(d'_1, k'_2) = 1$. Similarly, we have that $(d'_2, k'_1) = 1$. Thus the summation of two sides of (5) is

$$d_1 \sum_{t \in \mathbb{Z}_m, d'_1 | k'_2 t - n} 1 + d_2 \sum_{t \in \mathbb{Z}_m, d'_2 | k'_1 t - n} 1 = d_1 \sum_{t \in \mathbb{Z}_m, d'_1 | t} 1 + d_2 \sum_{t \in \mathbb{Z}_m, d'_2 | t} 1 = C(m, k_1, k_2)$$

(say). By (5) we have

$$d_1 \sum_{\substack{a \in A \\ d'_1 | k'_2 a - n}} 1 + d_2 \sum_{\substack{a \in A \\ d'_2 | k'_1 a - n}} 1 = \frac{1}{2} C(m, k_1, k_2) \quad (6)$$

for all integers n . In particular,

$$d_1 \sum_{\substack{a \in A \\ d'_1 | k'_2 a - d'_1 n}} 1 + d_2 \sum_{\substack{a \in A \\ d'_2 | k'_1 a - d'_1 n}} 1 = \frac{1}{2} C(m, k_1, k_2) \quad (7)$$

for all integers n . That is,

$$d_1 \sum_{\substack{a \in A \\ d'_1 | k'_2 a}} 1 + d_2 \sum_{\substack{a \in A \\ d'_2 | k'_1 a - d'_1 n}} 1 = \frac{1}{2} C(m, k_1, k_2) \quad (8)$$

for all integers n . Thus

$$\sum_{\substack{a \in A \\ d'_2 | k'_1 a - d'_1 n_1}} 1 = \sum_{\substack{a \in A \\ d'_2 | k'_1 a - d'_1 n_2}} 1 \quad (9)$$

for all integers n_1 and n_2 . Since $(d_1, d_2) = d_3$, we see that $(d'_1, d'_2) = 1$. By (9), $(d'_2, k'_1) = 1$, and $(d'_1, d'_2) = 1$, we have

$$\sum_{\substack{a \in A \\ d'_2 | a - u_1}} 1 = \sum_{\substack{a \in A \\ d'_2 | a - u_2}} 1 \quad (10)$$

for all integers u_1 and u_2 . So A is uniformly distributed modulo d'_2 . Similarly, A is uniformly distributed modulo d'_1 . Since $(d'_1, d'_2) = 1$, the set A is uniformly distributed modulo $d'_1 d'_2 = d_1 d_2 / d_3^2$.

Conversely, suppose that A is uniformly distributed modulo $d'_1 d'_2 = d_1 d_2 / d_3^2$. Then A is uniformly distributed modulo d'_1 . So

$$\sum_{\substack{a \in A \\ d'_1 | a - n}} 1 = \frac{|A|}{d'_1} = \frac{md_3}{2d_1} \quad (11)$$

for all integers n . Since $(k'_2, d'_1) = 1$, it follows that

$$\sum_{\substack{a \in A \\ d'_1 | k'_2 a - n}} 1 = \frac{md_3}{2d_1}$$

for all integers n . That is,

$$d_1 \sum_{\substack{a \in A \\ d_1 | k_2 a - d_3 n}} 1 = \frac{1}{2} md_3 \quad (12)$$

for all integers n . Similarly, we have

$$d_2 \sum_{\substack{a \in A \\ d_2 | k_1 a - d_3 n}} 1 = \frac{1}{2} md_3 \quad (13)$$

for all integers n . Since A is uniformly distributed modulo $d'_1 d'_2 = d_1 d_2 / d_3^2$, the set $B = \mathbb{Z}_m \setminus A$ is also uniformly distributed modulo $d'_1 d'_2 = d_1 d_2 / d_3^2$. Similarly, we have

$$d_1 \sum_{\substack{b \in B \\ d_1 | k_2 b - d_3 n}} 1 = \frac{1}{2} md_3 \quad \text{and} \quad d_2 \sum_{\substack{b \in B \\ d_2 | k_1 b - d_3 n}} 1 = \frac{1}{2} md_3 \quad (14)$$

for all integers n . By (12), (13), and (14), we see that (4) holds for all integers n . That is, (3) holds for all integers n with $d_3 | n$. For $d_3 \nmid n$, (3) holds trivially. So (3) holds for all $n \in \mathbb{Z}_m$. Therefore, (2) holds for all $n \in \mathbb{Z}_m$. \square

Proof of Corollary 1. Suppose that there exists a set $A \subseteq \mathbb{Z}_m$ such that $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. By Theorem 1, $|A| = m/2$ and A is uniformly distributed modulo $d_1 d_2 / d_3^2$, where $(k_1, m) = d_1$, $(k_2, m) = d_2$, and $(d_1, d_2) = d_3$. So m is even and

$$\frac{d_1 d_2}{d_3^2} \mid \frac{m}{2}.$$

That is,

$$\frac{2d_1d_2}{d_3^2} \Big| m. \quad (15)$$

If k_1 and k_2 have the different parities, say k_1 is even, then $v_2(d_1) = \min\{v_2(k_1), v_2(m)\}$ and $v_2(d_2) = v_2(d_3) = 0$. By (15) we have

$$1 + v_2(d_1) = 1 + v_2(d_1) + v_2(d_2) - 2v_2(d_3) \leq v_2(m).$$

So $v_2(k_1) = v_2(d_1) < v_2(m)$.

Conversely, suppose that m is even and one of (i) and (ii) of Corollary 1 holds.

Since $d_1 \mid m$, $d_2 \mid m$, and $(d_1/d_3, d_2/d_3) = 1$, it follows that $d_1d_2/d_3^2 \mid m$.

If k_1 and k_2 are both odd, then d_1/d_3 and d_2/d_3 are both odd. Noting that m is even, we have that $2d_1d_2/d_3^2 \mid m$.

If k_1 and k_2 are both even, say $v_2(k_1) \geq v_2(k_2)$, then

$$v_2(d_1) = \min\{v_2(k_1), v_2(m)\} \geq \min\{v_2(k_2), v_2(m)\} = v_2(d_2).$$

So $v_2(d_3) = v_2(d_2) \geq 1$ and

$$v_2\left(\frac{2d_1d_2}{d_3^2}\right) = 1 + v_2(d_1) + v_2(d_2) - 2v_2(d_3) \leq v_2(d_1) \leq v_2(m).$$

Noting that $d_1d_2/d_3^2 \mid m$, we have that $2d_1d_2/d_3^2 \mid m$.

If k_1 and k_2 have the different parities, say k_1 is even, then $v_2(d_1) = v_2(k_1) < v_2(m)$ and $v_2(d_2) = v_2(d_3) = 0$. Thus

$$v_2\left(\frac{2d_1d_2}{d_3^2}\right) = 1 + v_2(d_1) + v_2(d_2) - 2v_2(d_3) = 1 + v_2(d_1) \leq v_2(m).$$

By $d_1d_2/d_3^2 \mid m$, we have that $2d_1d_2/d_3^2 \mid m$.

Write $d = d_1d_2/d_3^2$. Then $2d \mid m$. Let

$$A = \bigcup_{i=1}^d \left\{ i + d\ell : \ell = 1, \dots, \frac{m}{2d} \right\}.$$

Then $|A| = m/2$ and A is uniformly distributed modulo d . By Theorem 1, $\hat{r}_{k_1, k_2}(A, n) = \hat{r}_{k_1, k_2}(\mathbb{Z}_m \setminus A, n)$ for all $n \in \mathbb{Z}_m$. \square

References

- [1] Y.-G. Chen, On the values of representation functions, *Sci. China Math.* 54 (2011) 1317-1331.
- [2] Y.-G. Chen and M. Tang, Partitions of nature numbers with the same representation functions, *J. Number Theory* 129 (2009) 2689-2695.
- [3] Y. G. Chen, B. Wang, *On additive properties of two special sequences*, *Acta Arith.* 110 (3) (2003) 299-303.
- [4] G. Dombi, *Additive properties of certain sets*, *Acta Arith.* 103 (2) (2002) 137-146.
- [5] V. F. Lev, *Reconstructing integer sets from their representation functions*, *Electron. J. Combin.* 11 (2004) R78.
- [6] C. Sándor, *Partitions of natural numbers and their representation functions*, *Integers* 4 (2004) A18.
- [7] M. Tang, *Partitions of the set of natural numbers and their representation functions*, *Discrete Math.* 308 (2008) 2614-2616.
- [8] Q. -H. Yang, F. -J. Chen, *Partitions of \mathbb{Z}_m with the same representation functions*, *Australas. J. Combin.* 53 (2012) 257-262.
- [9] Q. -H. Yang, Y. -G. Chen, *Partitions of natural numbers with the same weighted representation functions*, *J. Number Theory* 132 (2012) 3047-3055.